

# Fractional Action-Like Variational Problems\*

Rami Ahmad El-Nabulsi

Plasma Application Laboratory

Department of Nuclear and Energy Engineering

Faculty of Mechanical, Energy and Production Engineering

Cheju National University

Ara-dong 1, Jeju 690-756, South Korea

nabulsi.ahmadrami@yahoo.fr

Delfim F. M. Torres

Centre for Research on Optimization and Control

Department of Mathematics, University of Aveiro

Campus Universitário de Santiago

3810-193 Aveiro, Portugal

delfim@ua.pt

## Abstract

Fractional action-like variational problems have recently gained importance in studying dynamics of nonconservative systems. In this note we address multi-dimensional fractional action-like problems of the calculus of variations.

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# 1 Introduction

Fractional calculus (FC) represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [2, 20, 21, 24, 25, 26, 27, 30, 31, 32]. Although various fields of application of fractional derivatives and integrals are already well established, some others have just started. An example of the later is the study of fractional problems of the calculus of variations and respective Euler-Lagrange type equations [1, 4, 5, 6, 14].

During last years, fractional variational principles (FVP) were proposed and several applications given. Different methods were used to obtain the fractional Euler-Lagrange equations and the corresponding Hamiltonian canonical equations [1, 3, 4, 5, 6, 14, 22, 23, 28, 29, 33]. The existence of several different FVP and the need for still more elaboration on the subject, both for classical and quantized systems, is partially explained by the nonlocal nature of the fractional differential operators used to describe the dynamics, and the form of the corresponding adjoint operators. Another reason is the existence of many different fractional integral operators, including the ones of Grünwald-Letnikov, Caputo, Riesz, and Riemann-Liouville. The Riemann-Liouville operator is one of the most frequently used when fractional integration is performed [21, 30].

Recently, the first author has proposed a new one-dimensional (1D) approach, known as the fractional action-like variational approach (FALVA), in order to model nonconservative dynamical systems [7, 8]. In FALVA the fractional time integral introduces (only) one real parameter  $\alpha$ , and the derived Euler-Lagrange equations are simpler and similar to the standard ones. The novelty in the Euler-Lagrange equations is the presence of a fractional generalized external force acting on the system [7, 8, 9]. In particular, no fractional derivatives appear in the derived equations. The conjugate momentum, the Hamiltonian and Hamilton's equations are shown to depend on the fractional order of integration  $\alpha$ . Constants of motion for fractional action-like variational problems were discussed in [15, 16] (see also [17, 18]); FALVA problems with higher-order derivatives are studied in [16, 19]; and some encouraging results obtained and discussed in [10, 11, 12, 13, 14]. Here we are interested to generalize FALVA for multiple fractional action-like integrals of the calculus of variations (multi-dimensional fractional action-like

variational problems).

The text is organized as follows: in Sect. 2 we review the basic concepts of 1D-FALVA; the new extensions are found in Sect. 3. Since various applications of FC are based on replacing the time derivative in dynamical equations with a fractional derivative, in §3.1 we introduce the double-weighted FVP with the recent fractional derivatives of J. Cresson [6]. In §3.2 the FVP is given in general form, for the arbitrary  $N$ -dimensional case. Sect. 4 is dedicated to conclusions and future perspectives.

## A Note on the notation used

In order to be clear when  $f(t)$  stands for a function or the value of the function at a point  $t$ , we denote the function by  $t \rightarrow f(t)$  and the value of the function at  $t$  by  $f(t)$ . In the notation  $t \rightarrow f(t)$ ,  $t$  is a dummy variable. For instance, in the beginning of §2.2 we write  $(\dot{q}, q, \tau) \rightarrow L(\dot{q}, q, \tau)$ , which represents a function of three variables. Exactly the same function can be written, for example, as  $(a, b, c) \rightarrow L(a, b, c)$  ( $\dot{q}, q, \tau$  or  $a, b, c$  are here dummy variables). However, the dummy variables we choose in the paper are used consistently to fix the notation for the partial derivatives. For example, if we define the function as  $(a, b, c) \rightarrow L(a, b, c)$ , then we write  $\frac{\partial L}{\partial a}$  to denote the partial derivative of function  $L$  with respect to the first argument; if we define it as  $(\dot{q}, q, \tau) \rightarrow L(\dot{q}, q, \tau)$ , then the partial derivative of  $L$  with respect to the first argument is denoted by  $\frac{\partial L}{\partial \dot{q}}$ .

## 2 Brief Overview of 1D-FALVA

In this section we summarize the one-dimensional FALVA. The review consists of two cases: absence of fractional derivatives (§2.1); and presence of Riemann-Liouville fractional derivatives in the sense of Cresson [6] (§2.2). The reader is referred respectively to [7, 8] and [14] for more details.

### 2.1 Absence of fractional derivatives

Consider a smooth  $n$ -dimensional manifold  $M$  (configuration space) and denote by  $L : TM \times M \times \mathbb{R} \rightarrow \mathbb{R}$  the smooth Lagrangian function. For any smooth path  $q : [a, b] \rightarrow M$  satisfying fixed boundary conditions  $q(a) = q_a$

and  $q(b) = q_b$ , we define the fractional action integral by

$$S^\alpha [q] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t L(\dot{q}(\tau), q(\tau), \tau) (t - \tau)^{\alpha-1} d\tau, \quad (1)$$

where  $\Gamma$  is the Euler gamma function,  $\dot{q} = \frac{dq}{d\tau}$ ,  $0 < \alpha < 1$ ,  $\tau \in (a, t)$  is the intrinsic time and  $t \in [a, b]$  is the observer time. For a discussion of the importance to consider a multi-time formalism, we refer the reader to [34, 35]. The Lagrangian  $L(\dot{q}, q, \tau)$  is weighted by  $(t - \tau)^{\alpha-1}/\Gamma(\alpha)$ , and one can write the smooth action integral  $S^\alpha [q] (t)$  as  $\int_a^t L(\dot{q}(\tau), q(\tau), \tau) dg_t(\tau)$ , with an appropriate time smeared measure  $dg_t(\tau)$  on the time interval  $[a, t]$ , or as the Riemann-Liouville operator applied to the Lagrangian  $L(q'(t), q(t), t)$  [7, 8]. Here we just note that when  $\alpha \rightarrow 1$  the functional  $S^\alpha [q] (b)$  tends to the classical action integral of the calculus of variations:

$$\lim_{\alpha \rightarrow 1} S^\alpha [q] (b) = \int_a^b L(\dot{q}(\tau), q(\tau), \tau) d\tau.$$

Functionals of type (1) appear naturally in mathematical economy, where they are used for describing discounting economical dynamics; and in the theory of dynamical systems, for describing nonlinear dissipative structures.

The Euler-Lagrange equations associated with the fractional action integral (1) take the following form: for all  $t \in [a, b]$ ,

$$\frac{\partial L}{\partial q_i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{1 - \alpha}{t - \tau} \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n, \quad (2)$$

where the partial derivatives of  $L$  are evaluated at  $(\dot{q}(\tau), q(\tau), \tau)$ ,  $\tau \in (a, t)$ . The fractional term  $\frac{\alpha-1}{\tau-t} \frac{\partial L}{\partial \dot{q}_i}$  has a meaning in classical, quantum or relativistic settings, being responsible for adding a time-dependent damping coefficient into the dynamical equations. If we denote by  $R = (1 - \alpha)L/(t - \tau)$  the fractional Rayleigh dissipation function and by  $E$  the Euler-Lagrange operator, i.e.  $E = \frac{\partial}{\partial q} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{q}}$ , then equations (2) take the form  $E(L) = \frac{\partial R}{\partial \dot{q}}$ . Extremals are defined as solutions of the fractional Euler-Lagrange equations (FELE)  $E(L) = \frac{\partial R}{\partial \dot{q}}$ .

## 2.2 Presence of fractional derivatives

For any smooth path  $q : [a, b] \rightarrow M$  satisfying fixed boundary conditions  $q(a) = q_a$  and  $q(b) = q_b$ , let the fractional functional associated to  $(\dot{q}, q, \tau) \rightarrow L(\dot{q}, q, \tau)$  be now defined by

$$S_{\gamma}^{\alpha, \beta} [q] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t L(D_{\gamma}^{\alpha, \beta} q(\tau), q(\tau), \tau) (t - \tau)^{\alpha-1} d\tau, \quad (3)$$

where  $[a, t] \subseteq [a, b] \subset \mathbb{R}$ , and  $D_{\gamma}^{\alpha, \beta}$  denotes the Riemann-Liouville derivative of order  $(\alpha, \beta)$  as defined by J. Cresson in [6]: for all  $a, t \in \mathbb{R}$ ,  $a < t$ , the fractional derivative operator of order  $(\alpha, \beta)$ ,  $0 < \alpha, \beta < 1$ , is given by

$$D_{\gamma}^{\alpha, \beta} = \frac{1}{2} [D_{a+}^{\alpha} - D_{t-}^{\beta}] + \frac{i\gamma}{2} [D_{a+}^{\alpha} + D_{t-}^{\beta}], \quad (4)$$

with  $\gamma \in \mathbb{C}$ ,  $i = \sqrt{-1}$ ,  $D_{a+}^{\alpha}$  and  $D_{t-}^{\beta}$  the usual left and right Riemann-Liouville fractional derivatives of order  $0 < \alpha, \beta < 1$  [30],

$$D_{a+}^{\alpha} f(\theta) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\theta} \int_a^{\theta} f(\tau) (\theta - \tau)^{-\alpha} d\tau,$$

$$D_{t-}^{\beta} f(\theta) = \frac{1}{\Gamma(1-\beta)} \left( -\frac{d}{d\theta} \int_{\theta}^t f(\tau) (\tau - \theta)^{-\beta} d\tau \right),$$

$\theta \in [a, t]$ . For more details we refer the reader to [6]. Here we just note that for  $\gamma = i$  one has  $D_{\gamma}^{\alpha, \beta} = -D_{t-}^{\beta}$ ; for  $\gamma = -i$ ,  $D_{\gamma}^{\alpha, \beta} = D_{a+}^{\alpha}$ ; for  $\alpha \rightarrow 1$  and  $\beta \rightarrow 1$  one obtains  $D_{\gamma}^{\alpha, \beta} = \frac{d}{dt}$  and (3) is reduced to the classical functional of the calculus of variations. In [14] it is proved that the FELE correspondent to (3) is given by

$$\frac{\partial L}{\partial q} - D_{-\gamma}^{\beta, \alpha} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \dot{q}} \quad (5)$$

$\forall \tau \in (a, t)$ , where the partial derivatives of  $L$  are evaluated at  $(D_{\gamma}^{\alpha, \beta} q(\tau), q(\tau), \tau)$ .

In the next section we generalize (5) for FALVA problems with multiple integrals.

### 3 Main Results

More generally, we discuss here the multi-dimensional fractional variational problem or ND-FALVA ( $N$ -dimensional FALVA). We first start by double weighted fractional integrals, i.e. by the 2D-FALVA.

#### 3.1 Double-Weighted Fractional Variational Principles (2D-FALVA)

Similar to Sect. 2, we denote by  $M$  a smooth  $n$ -dimensional manifold; the admissible paths are smooth functions  $q : \Omega \subset \mathbb{R}^2 \rightarrow M$  satisfying fixed Dirichlet boundary conditions on  $\partial\Omega$ ; the Lagrangian function  $(q_x, q_y, q, x, y) \rightarrow L(q_x, q_y, q, x, y)$  is supposed to be sufficiently smooth with respect to all its arguments;  $\alpha$  and  $\beta$  are two real parameters taking values on the interval  $(0, 1)$ .

**Definition 1.** The 2D-FALVA action integral is defined by

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_{\Omega(\xi, \lambda)} L(D_{\gamma; x}^{\alpha, \delta} q, D_{\gamma; y}^{\beta, \chi} q, q, x, y) (\xi - x)^{\alpha-1} (\lambda - y)^{\beta-1} dx dy, \quad (6)$$

where  $\xi$  and  $\lambda$  are the observer times,  $(\xi, \lambda) \in \Omega$ ;  $x$  and  $y$  are the intrinsic times,  $(x, y) \in \Omega(\xi, \lambda) \subseteq \Omega$ ,  $x \neq \xi$  and  $y \neq \lambda$ ;  $q = q(x, y)$ ;  $D_{\gamma; x}^{\delta, \alpha}$  and  $D_{\gamma; y}^{\chi, \beta}$  are the fractional derivative operators (4), respectively of order  $(\delta, \alpha)$  with respect to  $x$  and of order  $(\chi, \beta)$  with respect to  $y$ . We denote (6) by  $S_{\gamma}^{\alpha, \beta, \delta, \chi}[q](\xi, \lambda)$ .

*Remark 1.* The classical multi-variable variational calculus has some limitations which a multi-time variational calculus, like the one we are proposing here, can successfully overcome. For a discussion of the importance to consider a multi-time formalism we refer the reader to the works of C. Udriste and his collaborators [34, 35].

The primary objective is to find functions  $q = q(x, y)$  that make the fractional action  $S_{\gamma}^{\alpha, \beta, \delta, \chi}[q](\xi, \lambda)$  stationary for every  $(\xi, \lambda) \in \Omega$ .

**Theorem 1.** *Given a smooth Lagrangian  $(q_x, q_y, q, x, y) \rightarrow L(q_x, q_y, q, x, y)$ , if  $q = q(x, y)$  makes the fractional action  $S_{\gamma}^{\alpha, \beta, \delta, \chi}[q](\xi, \lambda)$  defined by (6)*

stationary for every  $(\xi, \lambda) \in \Omega$ , then the following double-weighted Euler-Lagrange equation holds for every  $(x, y) \in \Omega(\xi, \lambda)$ :

$$\begin{aligned} & D_{-\gamma;x}^{\delta,\alpha} \left( \frac{\partial L(D_{\gamma;x}^{\alpha,\delta} q, D_{\gamma;y}^{\beta,\chi} q, q, x, y)}{\partial q_x} \right) + D_{-\gamma;y}^{\chi,\beta} \left( \frac{\partial L(D_{\gamma;x}^{\alpha,\delta} q, D_{\gamma;y}^{\beta,\chi} q, q, x, y)}{\partial q_y} \right) \\ & + \frac{1-\alpha}{\xi-x} \left( \frac{\partial L(D_{\gamma;x}^{\alpha,\delta} q, D_{\gamma;y}^{\beta,\chi} q, q, x, y)}{\partial q_x} \right) + \frac{1-\beta}{\lambda-y} \left( \frac{\partial L(D_{\gamma;x}^{\alpha,\delta} q, D_{\gamma;y}^{\beta,\chi} q, q, x, y)}{\partial q_y} \right) \\ & - \frac{\partial L(D_{\gamma;x}^{\alpha,\delta} q, D_{\gamma;y}^{\beta,\chi} q, q, x, y)}{\partial q} = 0. \quad (7) \end{aligned}$$

*Proof.* Let  $q$  be a stationary solution,  $h \ll 1$  a small real parameter, and  $w(x, y)$  an admissible variation, i.e. an arbitrary smooth function satisfying  $w(x, y) = 0$  for all  $(x, y) \in \partial\Omega$  so that  $q + hw$  satisfies the given Dirichlet boundary conditions for all  $h$ . The fractional action  $S_{\gamma}^{\alpha,\beta,\delta,\chi}[q + hw]$  can be written as

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_{\Omega(\xi,\lambda)} L(D_{\gamma;x}^{\alpha,\delta} q + h D_{\gamma;x}^{\alpha,\delta} w, D_{\gamma;y}^{\beta,\chi} q + h D_{\gamma;y}^{\beta,\chi} w, q + hw, x, y) \\ & \times (\xi - x)^{\alpha-1} (\lambda - y)^{\beta-1} dx dy, \end{aligned}$$

and the stationary condition  $\frac{d}{dh} S_{\gamma}^{\alpha,\beta,\delta,\chi}[q + hw] \big|_{h=0} = 0$  gives

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_{\Omega(\xi,\lambda)} \left( w \frac{\partial L}{\partial q} + D_{\gamma;x}^{\alpha,\delta} w \frac{\partial L}{\partial q_x} + D_{\gamma;y}^{\beta,\chi} w \frac{\partial L}{\partial q_y} \right) \\ & \times (\xi - x)^{\alpha-1} (\lambda - y)^{\beta-1} dx dy = 0, \quad (8) \end{aligned}$$

where the partial derivatives of function  $(q_x, q_y, q, x, y) \rightarrow L(q_x, q_y, q, x, y)$  are evaluated at  $(D_{\gamma;x}^{\alpha,\delta} q(x, y), D_{\gamma;y}^{\beta,\chi} q(x, y), q(x, y), x, y)$ . Using integration by parts and Green's theorem, we know that

$$\begin{aligned} & \iint_{\Omega(\xi,\lambda)} \left( \frac{\partial P}{\partial \xi} G_1 + \frac{\partial P}{\partial \lambda} G_2 \right) d\bar{\xi} d\bar{\lambda} \\ & = \oint_{\partial\Omega} P (-G_2 d\bar{\xi} + G_1 d\bar{\lambda}) - \iint_{\Omega(\xi,\lambda)} \left( P \left( \frac{\partial G_1}{\partial \xi} + \frac{\partial G_2}{\partial \lambda} \right) \right) d\bar{\xi} d\bar{\lambda} \end{aligned}$$

for any smooth functions  $G_1$  and  $G_2$ , where

$$\begin{aligned} \Gamma(1+\alpha) \bar{\xi} &= \xi^{\alpha} - (\xi - x)^{\alpha}, \\ \Gamma(1+\alpha) \bar{\lambda} &= \lambda^{\beta} - (\lambda - y)^{\beta}. \end{aligned}$$

We conclude from (8) that

$$\begin{aligned}
& -\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_{\Omega(\xi,\lambda)} w \left[ (\xi-x)^{\alpha-1} (\lambda-y)^{\beta-1} \times \left( D_{\gamma;x}^{\alpha,\delta} \left( \frac{\partial L}{\partial q_x} \right) + D_{\gamma;y}^{\beta,\chi} \left( \frac{\partial L}{\partial q_y} \right) \right) \right. \\
& + (1-\alpha) \left( \frac{\partial L}{\partial q_x} \right) (\xi-x)^{\alpha-2} (\lambda-y)^{\beta-1} + (1-\beta) \left( \frac{\partial L}{\partial q_y} \right) (\xi-x)^{\alpha-1} (\lambda-y)^{\beta-2} \\
& \left. - \frac{\partial L}{\partial q} (\xi-x)^{\alpha-1} (\lambda-y)^{\beta-1} \right] dx dy = 0
\end{aligned}$$

and then, because of the arbitrariness of  $w(x, y)$  inside  $\Omega(\xi, \lambda)$ , it follows (7), which is the 2D-FELE.  $\square$

We expect that fractional variational problems involving multiple integrals may have important consequences in mechanical problems involving dissipative systems with infinitely many degrees of freedom.

### 3.2 N-Weighted Fractional Variational Principles (ND-FALVA)

All the arguments of §3.1 can be repeated, *mutatis mutandis*, to the  $N$ -dimensional situation when the admissible paths are smooth functions  $q : \Omega \subset \mathbb{R}^N \rightarrow M$  satisfying given Dirichlet boundary conditions on  $\partial\Omega$ .

**Definition 2.** Consider a smooth manifold  $M$  and let  $(q_{x_1}, \dots, q_{x_N}, q, x_1, \dots, x_N) \rightarrow L(q_{x_1}, \dots, q_{x_N}, q, x_1, \dots, x_N)$  be a sufficiently smooth Lagrangian function. The ND-FALVA functional is defined by

$$S_{\gamma}^{\alpha,\delta}[q](\xi) = \frac{1}{\prod_{i=1}^N \Gamma(\alpha_i)} \int \cdots \int_{\Omega(\xi)} L(\nabla_{\gamma}^{\alpha,\delta} q(x), q(x), x) \prod_{i=1}^N (\xi_i - x_i)^{\alpha_i-1} dx,$$

where  $x = (x_1, \dots, x_N)$  is the intrinsic time vector,  $\xi = (\xi_1, \dots, \xi_N) \in \Omega$  the observer time vector,  $x \in \Omega(\xi) \subseteq \Omega$  with  $x_i \neq \xi_i$  ( $i = 1, \dots, N$ ),  $dx = dx_1 \cdots dx_N$ ,  $\alpha = (\alpha_1, \dots, \alpha_N)$ ,  $\delta = (\delta_1, \dots, \delta_N)$ ,  $0 < \alpha_i < 1$  ( $i = 1, \dots, N$ ), and  $\nabla_{\gamma}^{\alpha,\delta} = (D_{\gamma;x_1}^{\alpha_1,\delta_1}, \dots, D_{\gamma;x_N}^{\alpha_N,\delta_N})$ .



**Theorem 2.** *The  $N$ -dimensional Euler-Lagrange equation associated to the fractional functional  $S_\gamma^{\alpha,\delta}[q](\xi)$ ,  $\xi \in \Omega$ , is given by*

$$\sum_{i=1}^N \left[ D_{-\gamma; x_i}^{\delta_i, \alpha_i} \left( \frac{\partial L}{\partial q_{x_i}} \right) + \frac{1 - \alpha_i}{\xi_i - x_i} \left( \frac{\partial L}{\partial q_{x_i}} \right) \right] - \frac{\partial L}{\partial q} = 0,$$

where all partial derivatives of  $L$  are evaluated at  $(\nabla_\gamma^{\alpha,\delta} q(x), q(x), x)$ ,  $x \in \Omega(\xi)$ .

## 4 Conclusions and Further Work

In this work we introduce the multi-dimensional FALVA setting and derive the corresponding multi-dimensional fractional Euler-Lagrange equations. Obtained Euler-Lagrange equations are enough complicated, and one expects solutions to be found using numerical techniques. A work in this direction is in progress. We expect that fractional variational problems involving multiple integrals will have important consequences in mechanical problems and optimal control theory involving dissipative systems with infinitely many degrees of freedom. A geometric formulation of the field equations for the fractional action-like variational formalism, in terms of multi-vector fields on tangent bundles, is under investigation.

In our paper, as well as in all previous works on fractional Euler-Lagrange equations we are aware of, it is assumed that at least one stationary point for the fractional functional exist. Euler-Lagrange equations are valid under this assumption. The question of obtaining conditions on the Lagrangian  $L$  assuring the existence of stationary trajectories is, to the best of our knowledge, a completely open question in the fractional setting. This is a pertinent question because even very simple Lagrangians, e.g.  $L = D_\gamma^{\alpha,\beta} q$ , fail to satisfy the hypotheses under which the Euler-Lagrange equations are valid.

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## References

- [1] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, *J. Math. Anal. Appl.* **272** (2002), no. 1, 368–379.
- [2] O. P. Agrawal, Application of fractional derivatives in thermal analysis of disk brakes, *Nonlinear Dynam.* **38** (2004), no. 1-2, 191–206.
- [3] D. Baleanu and T. Avkar, Lagrangians with linear velocities within Riemann-Liouville fractional derivatives, *Nuovo Cimento* **119** (2004), 73–79.
- [4] D. Baleanu, S. I. Muslih and E. M. Rabei, On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative, *Nonlinear Dynamics*, in press. DOI:10.1007/s11071-007-9296-0; arXiv:0708.1690v1 [math-ph]
- [5] D. Baleanu and J. J. Trujillo, On exact solutions of a class of fractional Euler-Lagrange equations, *Nonlinear Dynamics*, in press. DOI:10.1007/s11071-007-9281-7; arXiv:0708.1433v1 [math-ph]
- [6] J. Cresson, Fractional embedding of differential operators and Lagrangian systems, *J. Math. Phys.* **48** (2007), no. 3, 033504, 34 pp.
- [7] R. A. El-Nabulsi, A fractional approach of nonconservative Lagrangian dynamics, *Fizika A* **14** (2005), no. 4, 289–298.
- [8] R. A. El-Nabulsi, A fractional action-like variational approach of some classical, quantum and geometrical dynamics, *Int. J. Appl. Math.* **17** (2005), no. 3, 299–317.
- [9] R. A. El-Nabulsi, Some geometrical aspects of fractional nonconservative autonomous Lagrangian mechanics, *Int. J. Appl. Math. Stat.* **5** (2006), no. S06, 50–64.
- [10] R. A. El-Nabulsi, Fractional path integral and exotic vacuum for the free spinor field theory with Grassmann anticommuting variables, *EJTP, Electron. J. Theor. Phys.* **4** (2007), no. 15, 157–164.

- [11] R. A. El-Nabulsi, Some fractional geometrical aspects of weak field approximation and Schwarzschild spacetime, *Rom. Journ. Phys.* **52** (2007), no. 5-7, 705–715.
- [12] R. A. El-Nabulsi, Cosmology with a fractional action principle, *Rom. Rep. Phys.* **59** (2007), no. 3, 759–765.
- [13] R. A. El-Nabulsi, I. A. Dzenite and D. F. M. Torres, Fractional action functional in classical and quantum field theory, *Scientific Proceedings of Riga Technical University, Series—Computer Science, Boundary Field Problems, and Computer Simulation*, 48th thematic issue, 2006, pp. 189–197.
- [14] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order  $(\alpha, \beta)$ , *Math. Methods Appl. Sci.* **30** (2007), no. 15, 1931–1939.
- [15] G. S. F. Frederico and D. F. M. Torres, Constants of motion for fractional action-like variational problems, *Int. J. Appl. Math.* **19** (2006), no. 1, 97–104.
- [16] G. S. F. Frederico and D. F. M. Torres, Non-conservative Noether’s theorem for fractional action-like variational problems with intrinsic and observer times, *Int. J. Ecol. Econ. Stat.* **9** (2007), no. F07, 74–82.
- [17] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether’s theorem for fractional problems of the calculus of variations, *J. Math. Anal. Appl.* **334** (2007), no. 2, 834–846.
- [18] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, *Nonlinear Dynamics*, in press. DOI:10.1007/s11071-007-9309-z; arXiv:0711.0609v1 [math.OC]
- [19] G. S. F. Frederico and D. F. M. Torres, Necessary optimality conditions for fractional action-like problems with intrinsic and observer times, *WSEAS Transactions on Mathematics*. In Special Issue: Nonclassical Lagrangian Dynamics and Potential Maps (Guest Editor: C. Udriste), **7** (2008), no. 1, 16–22. arXiv:0712.0152v1 [math.OC]
- [20] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in *Fractals and fractional calculus in continuum mechanics (Udine, 1996)*, 223–276, Springer, Vienna.
- [21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

- [22] M. Klimek, Fractional sequential mechanics—models with symmetric fractional derivative, Czechoslovak J. Phys. **51** (2001), no. 12, 1348–1354.
- [23] M. Klimek, Lagrangian fractional mechanics—a noncommutative approach, Czechoslovak J. Phys. **55** (2005), no. 11, 1447–1453.
- [24] R. L. Magin, Fractional calculus in bioengineering, Part 1, Critic. Rev. in Biomed. Eng. **32** (2004), no. 1, 110 pp.
- [25] R. L. Magin, Fractional calculus in bioengineering, Part 2, Critic. Rev. in Biomed. Eng. **32** (2004), no. 2, 90 pp.
- [26] R. L. Magin, Fractional calculus in bioengineering, Part 3, **32** (2004), no. 3-4, 183 pp.
- [27] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974.
- [28] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E (3) **53** (1996), no. 2, 1890–1899.
- [29] F. Riewe, Mechanics with fractional derivatives, Phys. Rev. E (3) **55** (1997), no. 3, part B, 3581–3592.
- [30] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
- [31] B. Stanković, An equation with left and right fractional derivatives, Publ. Inst. Math. (Beograd) (N.S.) **80(94)** (2006), 259–272.
- [32] V. E. Tarasov, Fractional variations for dynamical systems: Hamilton and Lagrange approaches, J. Phys. A **39** (2006), no. 26, 8409–8425.
- [33] V. E. Tarasov, Fractional generalization of gradient and Hamiltonian systems, J. Phys. A **38** (2005), no. 26, 5929–5943.
- [34] C. Udriste and I. Duca, Periodical solutions of multi-time Hamilton equations, Analele Universitatii Bucuresti **55** (2005), no. 1, 179–188.
- [35] C. Udriste and I. Tevy, Multi-time Euler-Lagrange-Hamilton theory, WSEAS Transactions on Mathematics **6** (2007), no. 6, 701–709.